Chapter 2: First order ODE

Section 2.1: Linear equations: method of integrating factors:

This section deals w/ ODE of first order

$$\frac{dy}{dt} = f(t,y), \text{ where}$$

$f(t,y)$ is linear in y.

In other words, the ODE's of the form

$$\frac{dy}{dt} + P(t)\cdot y = g(t)$$

We start from an example:

Example: Solve the ODE:

$$\frac{dy}{dt} + \frac{1}{2}y = \frac{1}{2}e^{t/3}$$

The idea is to multiply the equation by a function $\mu(t)$, (not ~~yet~~ determined yet):

$$\mu(t)\, y' + \frac{1}{2}\mu(t)\, y = \frac{1}{2}e^{t/3}\cdot \mu(t), \text{ s.t.}$$

the LHS is the derivative of something.

Let's look at the following formula:

$$\frac{d}{dt}\left(\mu(t)\cdot y\right) = \mu(t)\, y' + \mu'(t)\cdot y$$

Thus we need to find a solution of the following:

$$\frac{d\mu(t)}{dt} = \frac{1}{2}\mu(t)$$

$$\Downarrow$$

$$\frac{1}{\mu}\cdot\mu' = \frac{1}{2} \Longrightarrow \ln|\mu| = \frac{1}{2}t + C.$$

$$\Downarrow$$

$$\mu(t) = C e^{\frac{1}{2}t}$$

~~We~~ This $\mu(t)$ is called an integrating factor, put this back to the equation

$$e^{t/2}y' + \frac{1}{2}e^{t/2}y = \frac{1}{2}e^{5t/6}$$

LHS: $$\left(y\cdot e^{t/2}\right)' = \frac{1}{2}e^{5t/6}$$

$$\Longrightarrow \quad e^{t/2}\cdot y = \frac{3}{5}e^{5t/6} + c.$$

$$\Longrightarrow \quad y = \frac{3}{5}e^{t/3} + c\cdot e^{-t/2}$$

In particular, we can solve the initial value problem that $y(0)=1$

$$y = \frac{3}{5}e^{t/3} + \frac{2}{5}e^{-t/2}.$$

The fist step of generalization:

~~dy~~ $y' + ay = g(t)$, where a is a given constant,

Then the integration factor is e^{at}:

$$e^{at} y' + a \cdot e^{at} \cdot y = e^{at} g(t)$$

$$\Longrightarrow \quad (y \cdot e^{at})' = e^{at} \cdot (g(t))$$

$$y = e^{-at} \int_{t_0}^{t} e^{as} g(s)\, ds + c \cdot e^{-at}.$$

(See what happens if you vary t_0).

More generally:

$y' + p(t) \cdot y = g(t)$, where $p(t)$ and $g(t)$ are both functions of t. (i.e., a is not a constant anymore).

To find the ~~the~~ ~~in~~ integrating factor, note that

$$\mu(t) \cdot y' + p(t) \cdot \mu(t) \cdot y = \mu(t) \cdot g(t),$$

To make the LHS to be $(\mu(t) \cdot y)'$

$\cancel{(y \cdot \mu(t))'}$,

we need

$$\mu'(t) = p(t) \cdot \mu(t)$$

④

This gives $\left(\ln|\mu(t)|\right)' = p(t)$.

$$\Rightarrow \ln|\mu(t)| = \int p(t)\,dt + k$$

$\Rightarrow$ we take $k=0$

$$\mu(t) = \exp\left(\int p(t)\,dt\right).$$

Then the original equation becomes

$$\cancel{\frac{d}{dt}}\left(\mu(t)\cdot y\right)' = \mu(t)\,g(t).$$

$$\Rightarrow \mu(t)\cdot y = \int \mu(t)\,g(t) + c.$$

$$\Rightarrow y = \frac{1}{\mu(t)}\left[\int_{t_0}^{t} \mu(s)\,g(s)\,ds + c\right].$$

Example: Solve the initial problem

$$t\cdot y' + 2y = 4t^2.$$

$$y(1) = 2.$$

We can turn the equation to

$$y' + \frac{2}{t}y = 4t^2$$

The integrating factor is

$$\mu(t) = \exp\left(\int \frac{2}{t} \right).$$

$$= \exp\left(2\ln|t| \right) = t^2.$$

And the equation becomes

$$t^2\left(y' + \frac{2}{t} y \right) = t^2 \cdot 4t.$$

$$\Rightarrow \quad (t^2 \cdot y)' = 4t^3.$$

$$t^2 y = \int 4t^3.$$

$$= t^4 + c.$$

$$\Rightarrow y = t^2 + \frac{c}{t^2}$$

w/ the initial condition $y(1) = 2 \Rightarrow c = 1.$

$$y = t^2 + \frac{1}{t^2}, \quad \underline{\underline{t > 0}}$$

Remark: (1). $y = t^2 + \frac{1}{t^2}$ for $t < 0$ is NOT part of the solution of this initial value problem

(2). This is an example ~~of~~ in which the solution fails to exist for some values of t.

Example: Solve the initial value problem

$$2y' + ty = 2$$
$$y(0) = 1.$$

We turn the equation to

$$y' + \frac{t}{2} y = 2.$$

The integrating factor is

$$\mu(t) = \exp\left(\int \frac{t}{2} dt\right).$$
$$= e^{\frac{t^2}{4}}$$

And the equation becomes

$$\left(y \cdot e^{t^2/4}\right) = 2 \cdot e^{t^2/4}$$
$$y(t) = e^{-t^2/4}\left(\int_0^t e^{s^2/4} ds + c\right).$$

The initial condition $y(0) = 1$ requires that

$$c = 1.$$

Remark: There is no simple expression for $\int_0^t e^{s^2/4} ds.$

Separable equations:

We will use ~~the~~ x to denote the independent variable in this section for reasons clear later.

The general first order equation is of the form

$$\frac{dy}{dx} = f(x,y).$$

There's no ~~universally~~ general method for solving the equation. We will consider a subclass of first order equations that can be solved by direct integration.

First we can write our equation in the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0.$$

Suppose M is a function of x only and N is a function of y only, then it becomes

$$M(x) + N(y)\frac{dy}{dx} = 0$$

Such an equation is called separable, we can write it in the differential form

$$M(x)\,dx + N(y)\,dy = 0.$$

We can solve the equation by integrating $M(x)$ and $N(y)$.

Example 1: $$\frac{dy}{dx} = \frac{x^2}{1-y^2}.$$

We can write this equation as

$$-x^2 + (1-y^2)\frac{dy}{dx} = 0.$$

$$\Downarrow$$

$$M(x) = -x^2, \quad N(y) = 1-y^2.$$

To solve the equation:

$$(1-y^2)\,dy = x^2\,dx.$$

We integrate both sides.

$$y - \frac{y^3}{3} = \frac{x^3}{3} + C.$$

$$\Rightarrow \quad -x^3 + 3y - \frac{y^3}{3} = 3c.$$

This gives the equation of an integral curve.

To generalize: Let H_1 and H_2 be any antiderivatives of M and N respectively, i.e.:

$$H_1'(x) = M(x), \quad H_2'(y) = N(y).$$

Then the equation

$$M(x) + N(y)\frac{dy}{dx} = 0 \qquad (*)$$

becomes

$$H_1'(x) + H_2'(y)\frac{dy}{dx} = 0$$

By the chain rule

$$\Rightarrow \frac{d}{dx}\Big(H_1(x) + H_2(y(x))\Big) = 0.$$

$$\Rightarrow H_1(x) + H_2(y) = C. \qquad (**)$$

Remark: (1). Any differentiable function $y = \phi(x)$ that satisfies above original equation $(*)$ must also satisfy $(**)$. In other words $(**)$ defines the solution ~~ex~~ implicitly. (~~they~~ This is why numerical methods apply here!)

(2). An initial condition $y(x_0) = y_0$ determines the constant C:

$$C = H_1(x_0) + H_2(y_0)$$

Example: Solve the initial value problem:

$$\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y-1)}, \quad y(0) = -1.$$

The differential equation can be written as

$$2(y-1)\,dy = (3x^2+4x+2)\,dx$$

Integrating both sides, we get.

$$y^2-2y = x^3+2x^2+2x+C. \quad \text{, where } c \text{ is an arbitrary constant.}$$

To determine c satisfying our initial condition $y(0)=-1$, we notice that

$$\cancel{0 = 1+2-2+C.}$$

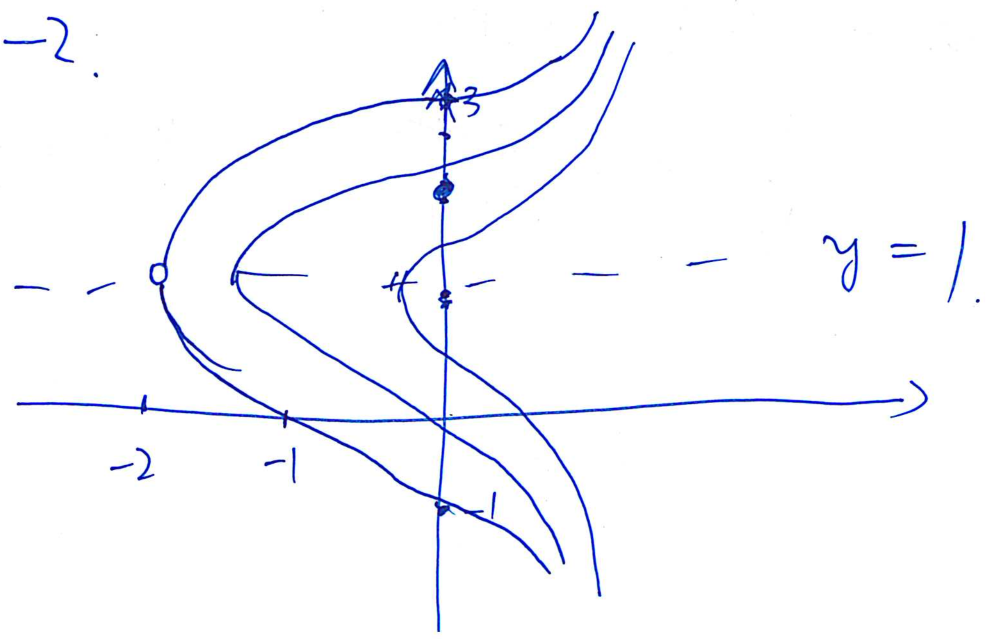
$$C = (-1)^2 - 2\cdot(-1) = 3.$$

To obtain the solution explicitly, we solve the equation for y in terms of x:

$$(y-1)^2 = x^3+2x^2+2x+4.$$

$$\Rightarrow y = 1 \pm \sqrt{x^3+2x^2+2x+4}.$$

We also need to determine the interval in which the quantity under the radical (i.e., x^3+2x^2+2x+4) is positive. This actually gives $x > -2$.



At time $t=0$, a tank contains Q_0 lb (pounds) of salt dissolved in 100 gal of water. Assume that water containing $\frac{1}{4}$ lb of salt/gal is entering the tank at a rate of r gal/min, ~~and that~~ the mixture is draining from the tank at the same rate. (1) Set up the initial value problem.

(2). Find the ~~amount of salt $Q(t)$ i tan~~ amount of salt $Q(t)$ in the tank at any time. In particular, find the limiting amount Q_L that is present as $t \to \infty$.

(3). ~~Find the time~~ Set $Q_0 = 2Q_L$, find the time T after which the salt level is within 2% of Q_L.

Solution: The variations in the amount of salt are due ~~to~~ ~~(1).~~ flows in and out of the tank:

Let $Q(t)$ denote the amount of salt at time t:

$$\frac{dQ}{dt} = \text{rate in} - \text{rate out}.$$

$$= \frac{r}{4} - \frac{rQ}{100}.$$

(~~Here we are using the fact that~~

The initial condition is $Q(0) = Q_0$.

$$\cancel{\frac{dQ}{dt} + \frac{rQ}{100}}$$

We can anticipate that eventually the mixture will be replaced by the mixture flowing in (w/ concentration $\frac{1}{4}$ lb/gal). So we might expect that ultimately the amount of salt in the tank would be close to $100 \times \frac{1}{4}$ lb/gal $= 25$ lb. Let's check this by solving the initial problem:

$$\frac{dQ}{dt} + \frac{rQ}{100} = \frac{r}{4}.$$

The integrating factor is $e^{rt/100}$

$$e^{rt/100} Q' + e^{rt/100} \cdot \frac{r}{100} \cdot Q = \frac{r}{4} \cdot e^{rt/100}$$

$$\Rightarrow \left(e^{rt/100} \cdot Q\right)' = \frac{r}{4} e^{rt/100}.$$

$$\Rightarrow e^{\frac{rt}{100}} Q(t) = \cancel{25} \int_0^t \frac{r}{4} e^{rs/100}\, ds + C.$$

$$= 25 e^{\frac{rt}{100}} + C.$$

$$\Rightarrow Q(t) = 25 + C \cdot e^{-\frac{rt}{100}}.$$

To satisfy the initial condition, we must take $C = Q_0 - 25$.

So the solution of the initial value problem is

$$Q(t) = 25 + (Q_0 - 25)\, e^{-rt/100}$$

We can see that

$$\lim_{t \to +\infty} Q(t) = 25.$$

Suppose now that $r = 3$, $Q_0 = 2Q_L = 50$, then

$$Q(t) = 25 + 25 \cdot e^{-0.03t} \qquad (*)$$

" Find the time T after which the salt level is within 2% of Q_L "

is saying the following:

$$Q_L = 25 \,\cancel{\text{gal}}\, \text{lb}, \quad Q_L \times 2\% = 0.5 \,\cancel{\text{gal}}\, \text{lb}.$$

We want to find the time T when $\cancel{Q(t)}$ $Q(T) = 25.5$.

Substituting $Q = 25.5$ in $(*)$

$$25 + 25 \cdot e^{-0.03T} = \cancel{0}\, 25.5$$

$$-0.03T = \ln\left(\frac{0.5}{25}\right)$$

$$T = \ln 50 / 0.03 \cong 130.4 \text{ (min)}.$$

Example 2: Escape Velocity:

A body of constant mass m is projected away from the earth in a direction perpendicular to the earth's surface w/ initial velocity. Assume that there is no air resistance. Now we can not assume the gravity is constant mg, because its distance to the surface of earth matters.

v: velocity,

x: distance to the surface of the earth,

gravity: $w(x) = -\dfrac{mgR^2}{(R+x)^2}$.

Equation of motion: $$m \cdot \frac{dv}{dt} = -\frac{mgR^2}{(R+x)}$$

$$v(0) = v_0$$

There are now too many variables: t, x, v.

The idea is to eliminate t from the equ. by thinking of x (the distance) as the independent variable. Then we need to express $\frac{dv}{dt}$ (acceleration) as ~~the~~ in terms of x by chain rule

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v \cdot \frac{dv}{dx}.$$

And the equation now becomes

$$v \cdot \frac{dv}{dx} = -\frac{g \cdot R^2}{(R+x)^2}$$

This equation is not linear but separable, we can integrate

$\cancel{v \cdot v'}$

both sides:

$$\frac{1}{2}v^2 = \frac{gR^2}{R+x} + C.$$

$x=0$ when $t=0$ (initial condition).

$\Rightarrow v = v_0$ when $x=0$.

$$\Rightarrow C = \frac{v_0^2}{2} - gR.$$

$$v = \pm\sqrt{v_0^2 - 2gR + \frac{2gR}{R+x}}$$

We must pick the correct sign $+$.

16.

The maximum altitude is attained when the velocity is 0.

$$\Rightarrow \quad v_0^2 - 2gR + \frac{2gR^2}{R+x} = 0.$$

The ~~hi~~ distance is then.

$$x+R = \left(\frac{2gR - v_0^2}{2gR^2}\right)^{-1} = \frac{2gR^2}{2gR - v_0^2}.$$

$$\Rightarrow \quad x = \frac{v_0^2 R}{2gR - v_0^2}$$

$$v_0 = \sqrt{2gR \frac{x}{R+x}}.$$

if $x \to \infty$, we get the escape velocity

$$v_e = \sqrt{2gR} \simeq 11.1 \ \text{Km/sec}.$$

Section 2.4:

We have now discussed a number of initial value problems, each of which had a solution and only one solution. So we can ask the

Question: Does every initial value problem have exactly one solution?

Theorem 2.4.1: If the functions p and g are <u>continuous</u> on an open interval $\alpha < t < \beta$ containing the point $t = t_0$, then there exists a <u>unique</u> function that satisfies the differential equation

$$y' + P(t)\,y = g(t)$$

for each $\alpha < t < \beta$. and the initial condition

$$y(t_0) = y_0,$$ where y_0 is an arbitrary prescribed initial value.

Proof: Explicit computation.

(linear 1st order ODE)

More generally, we have the following theorem:

Theorem 2.4.2: Let the function f and $\frac{\partial f}{\partial y}$ be continuous in some ~~interval~~ rectangle $\alpha < t < \beta$, $\gamma < y < \delta$ containing the point (t_0, y_0). Then in some interval $t_0 - h < t < t_0$ contained in $\alpha < t < \beta$, there is a unique solution $y = \phi(t)$ of the initial value problem

$$y' = f(t, y), \quad y(t_0) = y_0.$$

Observation: The hypotheses in this Theorem reduce to those in Theorem 2.4.1

$$y' = g(t) - p(t) \cdot y$$

$$f(t, y) = g(t) - p(t) \cdot y$$

$$\frac{\partial f}{\partial y} = -p(t), \quad \cancel{\frac{\partial f}{\partial t}}$$

The continuity of f and $\frac{\partial f}{\partial y}$ is equivalent to the continuity of p and g in this case.

Example: 1. ~~Use Theorem 2.4.1 to~~ find the interval in which the initial value problem

$$ty' + 2y = 4t^2 \qquad (\ast\ast)$$
$$y(1) = 2.$$

~~We~~ This is a linear equation:

$$y' + \frac{2}{t}y = 4t.$$

So $p(t) = \frac{2}{t}$, $g(t) = 4t$,

p is only continuous for $t<0$ or $t>0$.

So the initial value problem has a unique solution on the interval $0<t<\infty$.

Suppose we change the initial condition to $y(-1)=2$, then there is a unique solution for $t<0$.

Example 2: Consider the initial value problem

$$\frac{dy}{dx} = \frac{3x^2+4x+2}{2(y-1)}, \quad y(0) = -1.$$

We can apply Theorem 2.4.2: $f(x,y) = \dfrac{3x^2+4x+2}{2(y-1)}$.

f is continuous everywhere except on the line $y=1$.
Consequently, Theorem 2.4.2 gurantees that we can draw a rectangle containing the initial point $(0,-1)$ in which f and $\frac{\partial f}{\partial y}$ are continuous, s.t. the initial value problem in this rectangle has a unique solution.

Actually, this equation is separable and we can solve it explicitly:

$$y^2 - 2y = x^3 + 2x^2 + 2x + C.$$

$$\text{if } x=0,\ y=1 \implies C=-1.$$

$$y = 1 \pm \sqrt{x^3+2x^2+2x},$$

meaning that there are two functions ~~that~~ such ~~that~~.

(1). They satisfy the equation for $x > 0$.

(2). They satisfy the initial condition $y(0)=1$.

Example 3: Consider the initial value problem $y' = y^{1/3}$, $y(0) = 0$.

We can easily solve the equation: suppose $y \neq 0$.

then $y^{-\frac{1}{3}} y' = 1$.

$$\Downarrow$$

$$y^{-\frac{1}{3}} dy = dt.$$

$$\frac{3}{2} y^{\frac{2}{3}} = t + C.$$

$$y = \left(\frac{2}{3}(t+C)\right)^{3/2}.$$

The initial condition is satisfied if $C = 0$.

$$\Longrightarrow \quad y = \left(\frac{2}{3}t\right)^{\frac{3}{2}}, \quad t \geq 0.$$

However the function

① $y = -\left(\frac{2}{3}t\right)^{\frac{3}{2}}$

② $y \equiv 0$

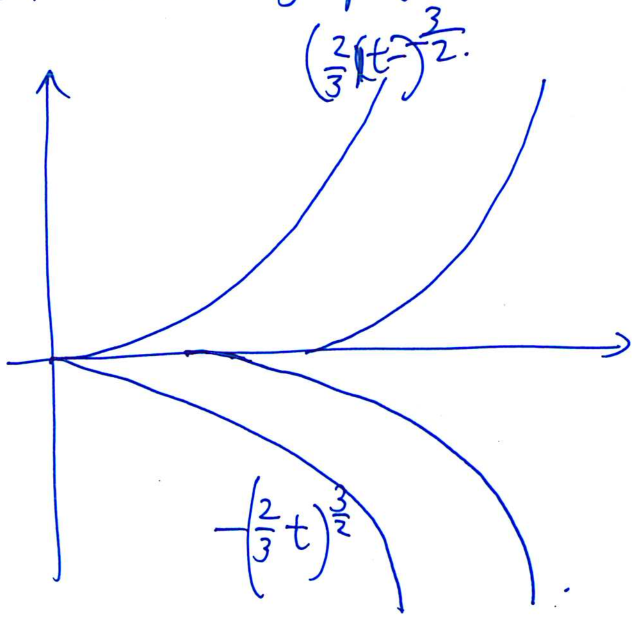
all satisfy the initial value problem.

More generally

$$y = \begin{cases} 0 & \text{for } 0 \le t \le t_0 \\ \pm\left[\frac{2}{3}(t-t_0)\right]^{3/2}, & \text{for } t \ge t_0 \end{cases}$$

are all solutions of this ~~equation~~ initial value problem.

We can draw a graph:



We can ask why ~~this~~ the theorem of uniqueness fails:

the function $f(x,y)$ in theorem 2.4.2 is

$$f(x,y) = y^{\frac{1}{3}} \Rightarrow \frac{\partial f}{\partial y} = \frac{1}{3} y^{-\frac{2}{3}}$$

so you can never apply theorem 2.4.2 if the initial point (x_0, y_0) is on the x-axis, i.e., $y_0 = 0$